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Equal partial revivals of wavepackets in long time scales

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Abstract. In terms of the action-angle Wigner function, we analyse the effects of energy spectrum anharmonicity on the evolutions of wavepackets in long time scales. The formulation shows a close connection between classical periodic motions of phase space points and the quantum (full and partial) revivals of wavepackets. This enables us to find the conditions of (full and partial) revivals in an intuitive way. For one-dimensional cases, we obtain analytic solutions for times of equal partial revivals, i.e. an initial wavepacket splits equally into several small parts in long time scales. Numerical results for the JC model confirm the theoretical predictions.

1. Introduction

Temporal behaviour of an initially prepared wavepacket have been extensively investigated both experimentally and theoretically [1–8]. These studies substantially deepen our insight into the fundamental problem of transition from quantum to classical mechanics [9,10]. Recent experimental advances in quantum optics and atomic physics [11–16] add new interest to this problem. For example, by ultrashort laser pulse, one can generate and monitor highly localized atomic Rydberg wavepackets which are a coherent superposition of highly excited states. This provides valuable information about structures of atoms. On the other hand, in cavity quantum electrodynamic (QED) experiments, an electromagnetic field is usually in wavepacket form, namely a field-coherent state. In high-Q cavities, one is able to study the long-time behaviour of a single mode of field wavepacket interacting with one atom, which is a powerful way to explore and test the fundamental model of field–atom interaction.

During evolution, a wavepacket often resumes its initial form repeatedly. In a short time scale, a wavepacket generally exhibits an overall motion along a classical orbit of phase space while its shape becomes deformed. This behaviour is similar to the corresponding classical Liouville density. This classical behaviour, however, stops when the discreteness of the energy spectrum takes into effect. Then quantum behaviour substantially deviates from its classical counterpart. In fact, there is only a finite number of frequencies $\{\omega_i\}$ effectively involved in the quantum motion. When every involved frequency's phase shift $\omega_i t$ becomes approximately the same (in the sense of mod 2π) with each other at time t, the initial wavepacket revives. This kind of behaviour is a general property of bounded quantum systems. It is pronounced in semiclassical regions, where the involved energy spectrum is almost equally spaced.

Between two consecutive full revivals, there are partial revivals. In terms of phase space distribution functions, such as the Wigner function [17,18] or Husimi function [19], the initial wavepacket breaks into several small regular forms. Generally speaking, full and partial revivals of wavepackets are the result of anharmonicity of the involved frequencies. In time scales when only the first order of the anharmonic term takes effect, i.e. the phase shifts caused by high-order anharmonic terms are negligibly small, the behaviour of full and partial revivals are simple [2,20]. The time interval between two consecutive full revivals is a constant. And at a fraction of the interval, the initial wavepacket breaks equally into several small wavepackets, which are equally spaced along the invariant tori.

When higher orders of anharmonicity take effect in longer time scales, a wavepacket has more complicated forms of (full or partial) revivals. One needs more sophisticated skills, as shown in [1], to find out the relations between full revival times and higher orders of anharmonicity. Under the influence of high-order anharmonicity, the initial wavepacket may become several unequal parts at times of partial revivals, which usually locate irregularly along the invariant tori [4].

To investigate the effects of high-order anharmonicity on the behaviour of wavepackets, we employ the Wigner function in terms of action-angle variables. This discrete phase space representation exhibits an intuitive relation with the corresponding classical Liouville density, and manifests naturally the conditions of full and partial revivals. In section 2 we show that, in terms of the action-angle Wigner function, quantum density is distributed in some discrete tori. Thus, there are partial revivals when the distribution on each individual torus recovers its initial form, and its central point moves to one of several possible angles. In this formulation, full revivals are special cases of partial revivals. In section 3, we obtain analytical expressions of such partial revival times for one dimensional cases. Application to the Jaynes–Cummings (JC) model is discussed in section 4. Numerical results for large detuning cases confirm with theoretical prediction. Conclusions and some comments are presented in section 5.

2. Conditions of partial revivals of wavepackets in terms of action-angle Wigner function

For a classical integrable system with k degrees of freedom, there exist k constants of motion $\boldsymbol{I} = (I_1, \ldots, I_k)$ that can serve as canonical momenta, namely the action variables. Each action variable $\boldsymbol{I}$ corresponds to an invariant torus. The conjugate coordinates of the action variables, i.e. the angle variables $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$, label positions on the invariant torus. Many properties of an integrable system have analytical forms in terms of action-angle variables. For example, evolution of the Liouville density can be simply written as $\rho_L(\boldsymbol{I}, \boldsymbol{\theta}, t) = \rho_L(\boldsymbol{I}, \boldsymbol{\theta} - \boldsymbol{\omega}(I)t, 0)$, where $\boldsymbol{\omega}(I) = (\omega_1(I), \ldots, \omega_k(I))$ is the frequency of the torus associated with the action $\boldsymbol{I}$, $\omega_i(I) = \frac{\partial}{\partial I_i} H(\boldsymbol{I})$ with $H(\boldsymbol{I})$ being the Hamiltonian.

Correspondingly, the quantum counterpart of the above system has k good quantum numbers $\boldsymbol{n} = (n_1, \ldots, n_k)$. The eigenenergies and their corresponding eigenstates are functions of the good quantum numbers, $H|\phi_{\boldsymbol{n}}\rangle = E(\boldsymbol{n})|\phi_{\boldsymbol{n}}\rangle$, where H is the Hamiltonian. In semiclassical regions, the quantum numbers have relations to classical action variables in the sense of EBK quantization [21], $I_i = \alpha_i + \hbar n_i$, where α_i is a constant.

From this observation, we define the action-angle Wigner function as follows:

$$F(\boldsymbol{\theta}, \boldsymbol{n}/2; t) = \frac{1}{2\pi} \sum_{\boldsymbol{n}'} \mathrm{e}^{-\mathrm{i}\boldsymbol{n}'\cdot\boldsymbol{\theta}} \langle \Psi(t)|\phi_{(\boldsymbol{n}+\boldsymbol{n}')/2}\rangle \langle \phi_{(\boldsymbol{n}-\boldsymbol{n}')/2}|\Psi(t)\rangle \tag{1}$$

where $\boldsymbol{n}' \cdot \boldsymbol{\theta} = n'_1\theta_1 + \cdots + n'_k\theta_k$, n_i and n'_i are integers; the summation is over all possible integers $\{n_1, n_2, \ldots\}$, and if one of $(n_i \pm n'_i)/2$ is half-integer, we define $|\phi_{(\boldsymbol{n}\pm\boldsymbol{n}')/2}\rangle = 0$. For the case of harmonic oscillator, the action-angle Wigner function is also referred to as the number-phase Wigner function. Many authors employ it [22–24] to investigate the quantum phase problem [25].

$F(\boldsymbol{\theta}, \boldsymbol{n}/2; t)$ has similar properties as the usual Wigner function: (i) It contains all information of the density operator, i.e. the density operator can be reconstructed from the action-angle Wigner function

$$\hat{\rho}(t) = \sum_{\boldsymbol{n}} \int_0^{2\pi} F(\boldsymbol{\theta}, \boldsymbol{n}/2; t) \hat{A}(\boldsymbol{\theta}, \boldsymbol{n}/2)\, \mathrm{d}\theta \tag{2}$$

where

$$\hat{A}(\boldsymbol{\theta}, \boldsymbol{n}/2) = \sum_{\boldsymbol{n}'} \mathrm{e}^{-\mathrm{i}\boldsymbol{n}'\cdot\boldsymbol{\theta}} |\phi_{(\boldsymbol{n}+\boldsymbol{n}')/2}\rangle\langle\phi_{(\boldsymbol{n}-\boldsymbol{n}')/2}|. \tag{3}$$

Using $\hat{A}(\boldsymbol{\theta}, \boldsymbol{n}/2)$, the action-angle Wigner function can be expressed as $F(\boldsymbol{\theta}, \boldsymbol{n}/2; t) = \frac{1}{2\pi}\,\mathrm{tr}(\hat{\rho}(t)\hat{A}(\boldsymbol{\theta}, \boldsymbol{n}/2))$. (ii) Integration of $F(\boldsymbol{\theta}, \boldsymbol{n}/2; t)$ over $\boldsymbol{\theta}$ results in the probability of the wavefunction located at state $|\phi_{\boldsymbol{n}/2}\rangle$ if $\boldsymbol{n}/2$ is an integer vector, i.e. all of its components, $n_1, n_2, \ldots$, are even-integer. However, if one of $n_i/2$ is half-integer, the integration vanishes. (iii) In the classical limit, the equation of motion of $f(\boldsymbol{\theta}, \boldsymbol{I}; t) = f(\boldsymbol{\theta}, \alpha + \hbar\boldsymbol{n}/2; t) = F(\boldsymbol{\theta}, \boldsymbol{n}/2; t)$ becomes the Liouville equation in terms of action-angle variables. (iv) In semiclassical regions, the expansion coefficients of a wavepacket, $\Psi_{\boldsymbol{n}}(0) = \langle\phi_{\boldsymbol{n}}|\Psi(0)\rangle$, is effectively non-zero only in a narrow region round $\boldsymbol{n}_0$. In this narrow region, the phase of $\Psi_{\boldsymbol{n}}(0)$ can be approximated by its Taylor expansion up to the first order, i.e. a linear function of the quantum numbers [2]. Thus the density of the wavepacket has the form

$$\hat{\rho}(0) \approx \sum_{\boldsymbol{m},\boldsymbol{n}} f_{\boldsymbol{m}} f_{\boldsymbol{n}} \mathrm{e}^{\mathrm{i}\boldsymbol{\delta}(\boldsymbol{n}_0)\cdot(\boldsymbol{m}-\boldsymbol{n})} |\phi_{\boldsymbol{m}}\rangle\langle\phi_{\boldsymbol{n}}| \tag{4}$$

where $f_{\boldsymbol{n}} = |\Psi_{\boldsymbol{n}}(0)|$, $\Psi_{\boldsymbol{n}}(0) = f_{\boldsymbol{n}} \mathrm{e}^{\mathrm{i}\Theta(\boldsymbol{n})}$, and $\delta_i(\boldsymbol{n}) = \frac{\partial}{\partial n_i}\Theta(\boldsymbol{n})$. From the Fourier transformation theory one sees that, in the action-angle phase space, the action-angle Wigner function of a wavepacket is centred at $(\delta(\boldsymbol{n}_0), \boldsymbol{n}_0)$, where $\boldsymbol{n}_0$ is the central point of $f_{\boldsymbol{n}}$ ($f_{\boldsymbol{n}}$ has maximum value at $\boldsymbol{n}_0$). (v) From (2), one can transform the action-angle Wigner function into other kinds of density distribution functions. For example, one obtains the Husimi function from the action-angle Wigner function by a smoothing procedure, and the smooth function is the action-angle Wigner function of the Gaussian wavepacket $F_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{\theta}, \boldsymbol{n})$,

$$\rho_H(\boldsymbol{q}, \boldsymbol{p}) = 2\pi \sum_{\boldsymbol{n}} \int_0^{2\pi} F(\boldsymbol{\theta}, \boldsymbol{n}) F_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{\theta}, \boldsymbol{n})\, \mathrm{d}\theta. \tag{5}$$

Here $(\boldsymbol{q}, \boldsymbol{p})$ is the central point of the Gaussian wavepacket. From the above properties and noting that $F(\boldsymbol{\theta}, \boldsymbol{n}/2; t)$ can be negative, we interpret $F(\boldsymbol{\theta}, \boldsymbol{n}/2; t)$ as quasi-probability of the wavefunction located at phase space point $(\boldsymbol{\theta}, \boldsymbol{n}/2)$ for all integers $\{n_1, n_2, \ldots\}$.

The action-angle Wigner function has two special properties. Firstly, it is periodic in the angle variables $F(\boldsymbol{\theta}, \boldsymbol{n}; t) = F(\boldsymbol{\theta} + 2\pi, \boldsymbol{n}; t)$; and secondly, it is distributed on discrete tori of the action-angle phase space. These two properties play important roles in our following discussions of (full and partial) revivals of wavepackets.

If the action-angle Wigner function of a density operator is initially distributed on one torus, i.e. it vanishes on other tori, then it will remain on this torus during the evolution. In fact, the density operator of an initial wavepacket is a linear combination of small components, and each component is only distributed on one torus:

$$\hat{\rho}(t) = \sum_{\boldsymbol{n}} \hat{\rho}_{\boldsymbol{n}}(t) \tag{6}$$

where

$$\hat{\rho}_{\boldsymbol{n}}(t) = \sum_{\boldsymbol{n}_1+\boldsymbol{n}_2=2\boldsymbol{n}} \langle\phi_{\boldsymbol{n}_1}|\hat{\rho}(t)|\phi_{\boldsymbol{n}_2}\rangle |\phi_{\boldsymbol{n}_1}\rangle\langle\phi_{\boldsymbol{n}_2}|. \tag{7}$$

Accordingly, the Wigner function can also be decomposed as

$$F(\boldsymbol{\theta}, \boldsymbol{n}; t) = \sum_m F_m(\boldsymbol{\theta}, \boldsymbol{n}; t) \tag{8}$$

where

$$F_m(\boldsymbol{\theta}, \boldsymbol{n}; t) = \frac{1}{2\pi} \operatorname{tr}(\hat{\rho}_m(t)\hat{A}(\boldsymbol{\theta}, \boldsymbol{n})). \tag{9}$$

$F_m(\boldsymbol{\theta}, \boldsymbol{n}; t) = \delta_{m,n} F_n(\boldsymbol{\theta}, \boldsymbol{n}; t)$ is only distributed on the torus associated with the quantum number $\boldsymbol{n}$. It is easy to see that evolution of the component $F_n(\boldsymbol{\theta}, \boldsymbol{n}; t)$ is restricted within its own torus.

The initial wavepacket revives (resumes it original form) at time $t = \tau$ if and only if each component resumes its initial form at a same angle $\boldsymbol{\Theta}(\tau)$, i.e. $F_n(\boldsymbol{\theta}, \boldsymbol{n}; \tau) = F_n(\boldsymbol{\theta} - \boldsymbol{\Theta}(\tau), \boldsymbol{n}; 0)$ with $\Theta(\tau)$ independent of $\boldsymbol{n}$. There are times when every $F_n(\boldsymbol{\theta}, \boldsymbol{n}; t)$ resumes its original shape, but their central angle points are located at different positions,

$$F_n(\boldsymbol{\theta}, \boldsymbol{n}; \tau) = F_n(\boldsymbol{\theta} - \boldsymbol{\Theta}(\boldsymbol{n}), \boldsymbol{n}; 0) \tag{10}$$

where $\boldsymbol{\Theta}(\boldsymbol{n})$ assumes one of several possible values. At these times, the original wavepacket becomes several small wavepackets, namely partial revivals of the initial wavepacket.

Since $\langle\phi_n|\Psi(t)\rangle = \exp(-\mathrm{i}E(\boldsymbol{n})t/\hbar)\langle\phi_n|\Psi(0)\rangle$, according to (7)–(9), equation (10) means that the partial revivals can occur at times when

$$[E((\boldsymbol{n}+\boldsymbol{n}')/2) - E((\boldsymbol{n}-\boldsymbol{n}')/2)]t/\hbar = \boldsymbol{\Theta}(\boldsymbol{n}/2) \cdot \boldsymbol{n}' \pmod{2\pi} \tag{11}$$

and

$$\boldsymbol{\Theta}(\boldsymbol{n}/2) \cdot \boldsymbol{n}' \pmod{2\pi} \in \{\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_M\} \tag{12}$$

where M is the number of small wavepackets that appear in the phase space. If $M = 1$, i.e. $\boldsymbol{\Theta}(\boldsymbol{n}/2)$ is independent of $\boldsymbol{n}$, (11) becomes the condition for full revivals of wavepackets, which is equivalent to that of [1].

According to our following discussions, equations (11) and (12) only predict 'equal' partial revivals, i.e. the initial wavepacket breaks equally into several small parts. However, we can obtain general conditions of partial revivals in the same way. To this end, we decompose a initial density operator as

$$\hat{\rho} = \tfrac{1}{2} \sum_{\boldsymbol{m},\boldsymbol{n}} \hat{\rho}_{mn} \tag{13}$$

where $\hat{\rho}_{mn} = |\phi_m\rangle\langle\phi_m|\hat{\rho}|\phi_n\rangle\langle\phi_n| + |\phi_n\rangle\langle\phi_n|\hat{\rho}|\phi_m\rangle\langle\phi_m|$. The action-angle Wigner function of $\hat{\rho}_{mn}$ is initially a cosine form,

$$\rho_{mn}(\boldsymbol{\theta}, \boldsymbol{l}, 0) = A_{mn}\cos[(m-n)(\boldsymbol{\theta} - \boldsymbol{\theta}_0)]\delta_{l,(m+n)/2}. \tag{14}$$

It is distributed only on the invariant torus associated with the quantum number $\boldsymbol{m} + \boldsymbol{n}$, and evolves in a way like a travelling wave,

$$\rho_{mn}(\boldsymbol{\theta}, \boldsymbol{l}, t) = A_{mn}\cos[(\boldsymbol{m} - \boldsymbol{n})(\boldsymbol{\theta} - \omega_{mn}t - \boldsymbol{\theta}_0)]\delta_{l,(m+n)/2} \tag{15}$$

where $\omega_{mn} \cdot (\boldsymbol{n} - \boldsymbol{m}) = [E(\boldsymbol{n}) - E(\boldsymbol{m})]/\hbar$. When all the phase shifts $\omega_{mn}t$ become the same angle (mod 2π), the initial wavepacket revives. Partial revivals occur at times when the phase shifts $\omega_{mn}t$ have several possible angles (mod 2π). i.e.

$$[E(\boldsymbol{n}) - E(\boldsymbol{m})]t/\hbar = \boldsymbol{\Theta}(\boldsymbol{n}, \boldsymbol{m}) \cdot (\boldsymbol{n} - \boldsymbol{m}) \pmod{2\pi} \tag{16}$$

where $\boldsymbol{\Theta}(\boldsymbol{n}, \boldsymbol{m}) \cdot (\boldsymbol{n} - \boldsymbol{m}) (\bmod 2\pi)$ has several possible values. The 'travelling waves' $\rho_{mn}(\boldsymbol{\theta}, \boldsymbol{l}, t)$ with the same phase shift form a small wavepacket. However, many of such partial revivals are 'unequal' ones, i.e. the initial wavepacket breaks into several unequal parts. From this general condition, one sees that at a fractional time of full revivals $(m/M)T$, there is a partial revival [4], where $m < M$ are two integers, and T is the full revival time.

3. 'Equal' partial revivals for the one-dimensional case

The conditions of equal partial revivals (11) and (12) have analytic solutions in one-dimensional cases. Here only one single quantum number is involved to expand the initial wavepacket in the eigenbasis of the Hamiltonian

$$|\Psi(0)\rangle \approx \sum_{n=n_{\min}}^{n_{\max}} \psi_n(0)|\phi_n\rangle \tag{17}$$

where the expansion coefficients of the wavepacket are effectively non-zero only in the region $n_{\min} < n < n_{\max}$. In this region, we assume that the spectrum $E(n)$ is approximately a quadratic function of the quantum number n, i.e. the coefficients of the Taylor expansion of $E(n)$ decrease rapidly for third and higher orders. Thus the spectrum can be expressed as

$$E(n) = \sum_{i=0}^{\infty} \omega_i (n - n_0)^i \tag{18}$$

with $\{\omega_i\}$ satisfying

$$|\omega_2| \gg |\omega_3| \gg \cdots \gg |\omega_m| \gg |\omega_{m+1}| \gg \cdots. \tag{19}$$

Here n_0 is the central point of the initial wavepacket in the eigenbasis, $|\psi_{n_0}| \geqslant |\psi_n|$, and $\omega_i = \frac{1}{i!}\frac{\partial^{(i)}}{\partial x^i}E(x)|_{x=n_0}$, $\omega_0 = E(n_0)$. In semiclassical regions, the effective Planck constant $\overline{h}$ is a small quantity, $\overline{h} \ll 1$, and $\omega_i \approx \overline{h}^i \frac{1}{i!}\frac{\partial^{(i)}}{\partial I^i}E(I)|_{I=I_0}$, with I being the action variable and $I_0 = \alpha + \overline{h}n_0$. Thus, equation (19) is usually satisfied in the semiclassical region.

Let $T_i = 2\pi/|\omega_i|$, thus $\{T_i\}$ form a cascade of time scales $T_2 \ll T_3 \ll \cdots$. At a time scale $(t/\overline{h}) < T_\lambda$, the higher-order terms $\omega_{\lambda+1}(n - n_0)^{\lambda+1}, \ldots$, can be neglected in the phase shifts $\delta_\lambda = E(n)t/\overline{h}$,

$$\delta_\lambda(k) = (t/\overline{h})E_\lambda(n) \approx (t/\overline{h}) \sum_{i=0}^{\lambda} \omega_i k^i \tag{20}$$

where $k = n - n_0$. In the one-dimensional case, (11) becomes

$$[E(n) - E(m)]t/\overline{h} = \Theta(m+n)(n-m) \pmod{2\pi}. \tag{21}$$

This condition requires that $\Theta(m+n) (\mathrm{mod}\, 2\pi)$ be an effectively linear function of $m+n$ (see the appendix):

$$[E(n) - E(m)]t/\overline{h} = [\Theta_0 + \beta(m+n)](n-m) \pmod{2\pi}. \tag{22}$$

The condition (12) demands that $\beta = 2\pi l/M$ with l, M being integers. Putting this into (21) and letting $m = n_0$, we obtain the condition for equal partial revivals in the one-dimensional case,

$$[E(n) - E(n_0)]t/\overline{h} = \theta_0(n - n_0) + \frac{2\pi l}{M}(n - n_0)^2 \pmod{2\pi} \tag{23}$$

where $\theta_0 = \Theta_0 + 2n_0\beta$ is an arbitrary real number.

From (20) and (23), the times of partial revivals can be worked out in a similar way to that used in [1]. To this end, we rewrite $E_\lambda(n) - E(n_0)$ of (20) as

$$E_\lambda(n) - E(n_0) = \sum_{i=1}^{\lambda} \zeta_i \prod_{j=0}^{i-1}(k - j) \tag{24}$$

where $k = n - n_0$ and ζ_i is linear combination of $\omega_i, \ldots, \omega_\lambda$,

$$\zeta_i = \sum_{l=i}^{\lambda} \xi_{il}\omega_l \tag{25}$$

where the coefficients $\{\xi_{il}\}$ are positive integers and $\xi_{ii} = 1$. Putting (24) into (23), it becomes

$$\sum_{i=3}^{\lambda} \zeta_i t/\hbar \prod_{j=0}^{i-1}(k-j) + \left(\zeta_2 t/\hbar - \frac{2\pi l}{M}\right) k(k-1) + \left(\zeta_1 t/\hbar - \theta_0 - \frac{2\pi l}{M}\right) k = 0 \pmod{2\pi}. \tag{26}$$

Thus the conditions for equal partial revivals become

$$(t/\hbar) i! \sum_{l=i}^{\lambda} \xi_{il}\omega_l = 0 \pmod{2\pi} \qquad (i = 3, \ldots, \lambda) \tag{27}$$

$$(t/\hbar) 2! \sum_{l=2}^{\lambda} \xi_{2l}\omega_l = 0 \pmod{2\pi/M}. \tag{28}$$

Solutions of (27) and (28) can be obtained in a recursive way. Without loss of generality, the times $t/\hbar = \tau_\lambda$ of partial revivals can be expressed as

$$\tau_\lambda = \tau^{(2)} + \cdots + \tau^{(\lambda)} \tag{29}$$

with $\tau^{(i)} \gg \tau^{(i-1)}$, namely $\tau^{(\lambda)}$ is the main part of τ_λ, and $\tau^{(\lambda-1)}$ is the first-order amendment to $\tau^{(\lambda)}$, $\tau^{(\lambda-2)}$ is the second-order amendment, and so on.

Note that the case of $i = \lambda$ in (27) contains only ω_λ, and the case of $i = \lambda - 1$ contains only $\omega_{\lambda-1}$ and ω_λ, and so on. From the property that $\omega_i \gg \omega_{i+1}$, we let $\tau^{(\lambda)}$ be the solution of the case of $i = \lambda$, it can be expressed as

$$\tau^{(\lambda)} = \frac{k_\lambda}{\lambda!}\frac{2\pi}{|\omega_\lambda|}. \tag{30}$$

Here the integer k_λ must satisfy $k_\lambda T_\lambda \ll T_{\lambda+1}$. The first-order amendment $\tau^{(\lambda-1)}$ is obtained from the requirement that $\tau^{(\lambda-1)} + \tau^{(\lambda)}$ is the solution of (27) for the case of $i = \lambda - 1$, thus we have

$$\tau^{(\lambda-1)} = \frac{k_{\lambda-1}}{(\lambda-1)!}\frac{2\pi}{|\zeta_{\lambda-1}|} - \frac{1}{(\lambda-1)!|\zeta_{\lambda-1}|} \cdot \mathrm{mod}[\tau^{(\lambda)}(\lambda-1)!|\zeta_{\lambda-1}|, 2\pi]. \tag{31}$$

Here $0 < \mathrm{mod}(x, y) < y$ is the remainder of x/y, the integer $k_{\lambda-1}$ must satisfies $k_{\lambda-1}T_{\lambda-1} \ll T_\lambda$. From the fact that $\zeta_{\lambda-1} \approx \omega_{\lambda-1}$, or $\tau^{(\lambda-1)}$ is within the scale of $T_{\lambda-1}$, thus $\tau^{(\lambda-1)} + \tau^{(\lambda)}$ also satisfies (27) with $i = \lambda$ since $\omega_\lambda \tau^{(\lambda-1)}$ is negligibly small. Similarly, $\tau^{(j)}$ is obtained from the requirement that $\tau^{(j)} + \cdots + \tau^{(\lambda)}$ being the solution of (27) for the case $i = j$:

$$\tau^{(j)} = \frac{k_j}{j!}\frac{2\pi}{|\zeta_j|} - \frac{1}{j!|\zeta_j|} \cdot \mathrm{mod}[j!|\zeta_j|(\tau^{(j+1)} + \cdots + \tau^{(\lambda)}), 2\pi]. \tag{32}$$

Here the requirement $\tau^{(j)} \ll \tau^{(j+1)}$ demands that the integer k_j satisfies $k_j T_j \ll T_{j+1}$. Note that $\zeta_j \approx \omega_j$, or $\hbar\tau^{(j)} \sim T_j$, thus $\tau^{(j)}$ is a small perturbation to $(\tau^{(j+1)} + \cdots + \tau^{(\lambda)})$ which is the solution for the cases $i = j + 1, \ldots, \lambda$. This means that $\tau^{(j)} + \cdots + \tau^{(\lambda)}$ is the solution of (27) for the cases of $i = j, \ldots, \lambda$.

Finally, $\tau^{(2)}$ is obtained from the requirement of $\tau_\lambda = \tau^{(2)} + \cdots + \tau^{(\lambda)}$ being the solution of (28):

$$\tau^{(2)} = \frac{k_2}{2!}\frac{2\pi}{M|\zeta_2|} - \frac{1}{2!|\zeta_2|} \cdot \mathrm{mod}[2!|\zeta_2|(\tau^{(3)} + \cdots + \tau^{(\lambda)}), 2\pi/M]. \tag{33}$$

Here the integer k_2 satisfies $k_2 T_2 \ll T_3$. Similar arguments show that τ_λ is also the solution of (27) for the cases $i = 3, \ldots, \lambda$.

When $M = 1$, the partial revival times expressed in (29)–(33) become the full revival times similarly to the case in [1].

The first partial revival occurs at the time scale $t \sim T_2$. In this time scale, the times when the initial wavepacket becomes M small wavepackets are

$$\tau_2 = \frac{k_2}{2!M}\frac{2\pi}{|\omega_2|}. \tag{34}$$

Here k_2 and M has no common factor. In the next time scale $t \sim T_3$, $\zeta_3 = \omega_3$, $\zeta_2 = \omega_2 + 3\omega_3$, the partial revivals happen at $\tau_3 = \tau^{(2)} + \tau^{(3)}$ with

$$\tau^{(3)} = \frac{k_3}{3!}\frac{2\pi}{|\omega_3|} \tag{35}$$

and from (33)

$$\tau^{(2)} = \frac{k_2}{2!M}\frac{2\pi}{|\omega_2 + 3\omega_3|} - \frac{1}{2!|\omega_2 + 3\omega_3|}\mathrm{mod}[2!\tau^{(3)}|\omega_2 + 3\omega_3|, 2\pi/M]. \tag{36}$$

Partial revivals described by (29)–(33) are 'equal' ones, i.e. the initial wavepacket breaks into several equal parts. This is evident from equations (22) and (23). The action-angle Wigner function recovers its initial form in every torus, and nearby tori move against each other by $2\pi/M$ in the angle direction, where M is the number of small wavepackets. Thus the density distribution on torus $l, l+M, l+2M, \ldots$ $(l = 1, 2, \ldots)$ forms a small wavepacket. The M small wavepackets are equally distributed along the invariant tori.

4. An example: partial revivals in the JC model

To illustrate the above discussions, we analyse partial revivals in the JC model [26]. This model describes a two-level atom interacting with a single mode of quantized radiation field. This solvable quantum system exhibits many fascinating quantum effects that can be tested experimentally [27,28], including partial revivals of initially coherent field states, also referred as emerging of Schrödinger cat states [29,30]. Here we consider a simple form of the JC model with the Hamiltonian

$$H = \hbar\omega a^+a + \sum_{i=0}^{1} e_i|e_i\rangle\langle e_i| + g(a|e_1\rangle\langle e_0| + a^+|e_0\rangle\langle e_1|) \tag{37}$$

where $|e_0\rangle$ and $|e_1\rangle$ are the ground and excited states of the two-level atom respectively; e_0 and e_1 are the two corresponding eigenenergies; a^+ and a are the creation and annihilation operators of the field with commutation relation $[a, a^+] = 1$ and the real number g is the coupling parameter. This integrable system has analytic solutions. The spectrum can be expressed as a function of two quantum numbers

$$E(n, s) = \hbar\omega n + e_0 + \Delta + (2s-1)g\sqrt{n + (\Delta/g)^2} \tag{38}$$

where $s = 0$ or 1, $n = 0, 1, 2, \ldots$, and $\Delta = (e_1 - e_0 - \hbar\omega)/2$ is the detuning factor. The corresponding eigenstates can be written as

$$|\phi(n, s)\rangle = \cos(\gamma_{n,s})|n, e_0\rangle + \sin(\gamma_{n,s})|n-1, e_1\rangle \tag{39}$$

where $\gamma_{n,1} = \gamma_{n,0} + \pi/2$; $|n, e_i\rangle = |n\rangle \otimes |e_i\rangle$, and $|n\rangle$ is the eigenvector of the number operator a^+a, $a^+a|n\rangle = n|n\rangle$. For a large detuning factor $\Delta \gg 1$, one can show that $\gamma_{n,0} \approx 0$, thus $|n, e_i\rangle$ is approximately the eigenfunction of the Hamiltonian.

Since the quantum number s has only two possible values 0 and 1, a wavefunction in eigenbasis $\{\phi(n, s)\}$ can be divided into two branches according to the quantum number s:

$$|\Psi(t)\rangle = |\Psi_0(t)\rangle + |\Psi_1(t)\rangle \tag{40}$$

where

$$|\Psi_s(t)\rangle = \sum_n \psi_{n,s}(t)|\phi(n,s)\rangle \qquad (s = 0, 1). \tag{41}$$

In this way, the evolution of the two-dimensional density operator $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$ can be treated as independent evolutions of four one-dimensional density operators

$$\hat{\rho}(t) = \hat{\rho}_{00}(t) + \hat{\rho}_{01}(t) + \hat{\rho}_{10}(t) + \hat{\rho}_{11}(t) \tag{42}$$

where $\hat{\rho}_{ij}(t) = |\Psi_i(t)\rangle\langle\Psi_j(t)|$ depends only on one quantum number n. Note that $2s-1 = \pm 1$, the nonlinear part of the spectrum $E(n,s)$ in (38) is the same $g\sqrt{n + (\Delta/g)^2}$ for different s. Thus the four density operators $\{\hat{\rho}_{ij}(t)\}$ have same full or partial revival periods. But the central points where the initial wavepacket revives or partially revives, which depend on the linear terms of the spectrum, are different. This means that at partial revival times when each one-dimensional wavepacket becomes M small wavepackets, there are $4M$ small wavepackets in the phase space distribution.

For an initial wavepacket centred at n_0, if $f_0 = \sqrt{n_0 + (\Delta/g)^2} \gg 1$, the coefficients of the energy spectrum's Taylor expansion satisfy condition (19). This can be achieved by either choosing a large detuning factor Δ, or preparing the initial wavepacket with a large mean photon number. Applying (29)–(36), we obtain the partial revival times for the JC model: In the time scale $t \sim T_2$, the initial wavepacket becomes M small wavepackets at times

$$\tau_2(k_2, M) = \frac{k_2}{M}\frac{4\pi f_0^3}{g}. \tag{43}$$

In the time scale $t \sim T_3$, the M small wavepackets appear at times

$$\tau_3(k_2, k_3, M) = \frac{k_3}{9}\frac{8\pi f_0^5}{g} + \frac{k_2}{M}\frac{8\pi f_0^5}{g(2f_0^2 - 3)} - \frac{4 f_0^5}{g(2f_0^2 - 3)} \mathrm{mod}\left[\frac{4k_3\pi f_0^2}{9}, \frac{2\pi}{M}\right]. \tag{44}$$

Here k_2, k_3 are integers and k_2 has no common divisor with M. For $M = 1$, (43) and (44) are the full revival times in these time scales.

The numerical calculation is performed for a large detuning case with the initial state chosen as a coherent field state and the atom in the ground state,

$$|\Psi(0)\rangle = |q_0, p_0\rangle \otimes |e_0\rangle \tag{45}$$

where $|q_0, p_0\rangle = \exp(z_0 a^+ - z_0^* a)|0\rangle$ with the complex number $z_0 = (q_0 + \mathrm{i} p_0)/\sqrt{2\hbar}$, and $|0\rangle$ being the vacuum field state. Figures 1 and 2 show the Husimi distributions of the reduced density operator that is traced over atomic space,

$$\rho_H(q, p) = \langle q, p|\hat{\rho}^{(f)}(t)|q, p\rangle \tag{46}$$

where

$$\hat{\rho}^{(f)}(t) = \sum_{s=0}^{1}\langle e_s|\hat{\rho}(t)|e_s\rangle. \tag{47}$$

The Husimi distribution of the density operator $\hat{\rho}(t)$ can be viewed as a superposition of four Husimi distributions of $\hat{\rho}_{ij}(t)$ $(i, j = 0, 1)$. But for the initial state (45), the contributions of $\hat{\rho}_{01}(t)$, $\hat{\rho}_{10}(t)$, and $\hat{\rho}_{11}(t)$ are very small in large detuning case. Thus the initial wavepacket (figure 1(*a*)) is virtually contributed by $\hat{\rho}_{00}(t)$, and the evolution can be treated as a one-dimensional case. The parameters for the system are chosen as $e_0 = 0$, $e_1 = 1$, $\omega = 100$, $g = 1$, and the effective Planck $\hbar$ constant is set to 1 (arbitrary units). This is a highly detuned case with detuning factor $\Delta = 49.5$. With this setting, the first three time scales are $T_1 = 624.189$, $T_2 = 6.160\,13 \times 10^6$ and $T_3 = 3.039\,71 \times 10^{10}$.

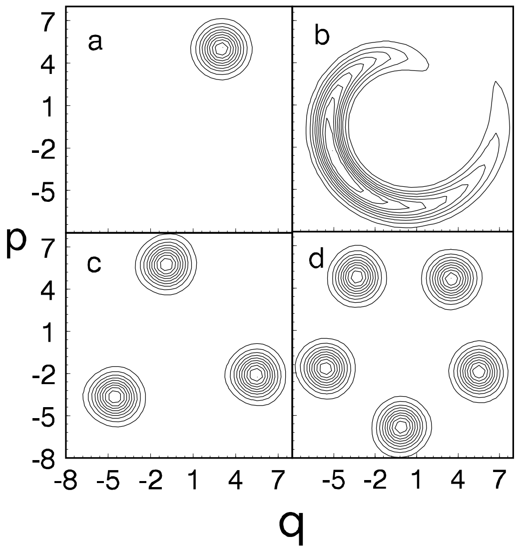

Figure 1. Contour plots of Husimi distribution of a wavepacket at (*a*) initial time $t = 0$, (*b*) $t = 153\,566 \approx \tau_2/20$, (*c*) $t = 615\,513 \approx \tau_2/3$, (*d*) $t = 615\,513 \approx \tau_2/5$. The central point of the Gausian wavepacket is initially located at $q_0 = 3$, $p_0 = 5$, and the effective Planck constant is set to $\hbar = 1$, (arbitrary units). The parameters for the Hamiltonian are as follows: $e_0 = 0$, $e_1 = 1$, $\omega = 100$, $g = 1$, (arbitrary units). This is a highly detuned case.

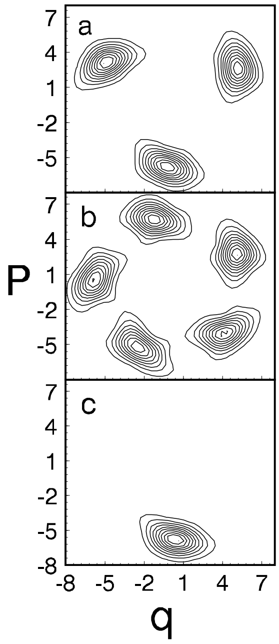

Figure 2. Contour plots of the Husimi distribution of the wavepacket in long time scales, (*a*) $t = 5.0773 \times 10^9 \approx \tau_3(2, 1, 3)$, (*b*) $t = 5.067\,32 \times 10^9 \approx \tau_3(2, 1, 5)$, (*c*) $t = 5.069\,79 \times 10^9 \approx \tau_3(2, 1, 1)$, where τ_3 is defined in (44).

In a short time scale $t \ll T_2$, the second- or higher-order terms of the spectrum's Taylor expansion can be neglected in the phase shift $E(n, s)t/\hbar$, i.e. it is approximately a linear function of the quantum number n, thus the evolution of the Husimi distribution of $\hat{\rho}_{00}(t)$ is similar to a classical particle moving along a classical orbit. It is easy to show that the frequency of classical motion of the wavepacket is $|\omega_1| = |\omega - g/\sqrt{n_0 + (\Delta/g)^2}|$, namely the wavepacket returns to its initial position after a period $T_1 = 2\pi/|\omega_1|$.

This kind of classical behaviour can only last for a short period of time. As time increases, the nonlinear terms of the spectrum take effect gradually in the phase shift $E(n, s)t/\hbar$, which causes the wavepacket to spread along the invariant tori of the phase space. The motion of the wavepacket in this period is an interplay between classical overall shift and spread of the wavepacket (figure 1(*b*)). The process continues until the wavepacket spreads over all possible tori.

In the time scale $t \sim T_2$, the density distribution of $\hat{\rho}_{00}$ resumes regular form at the time $t = \tau_2(k_2, M)$ of (43): the original density distribution becomes M small wavepackets, and they move along the classical orbits in a way similar to their initial distribution. Figures 1(*c*) and (*d*) show the Husimi distributions when the wavepacket becomes $M = 3$ and $M = 5$ small wavepackets respectively. This kind of behaviour repeats at $t = \tau_2(k_2, M)$ for each integer k_2 (k_2 has no common divisor with M).

In fact, within this time scale, the evolution of the action-angle Wigner function can be expressed in the same form as that of the classical Liouville density [20]:

$$F(\theta, n/2; t) = F(\theta - (\omega_1 + \omega_2 n)t, n; 0). \tag{48}$$

In this way, the evolution of $F(\theta, n/2; t)$ can be viewed as a result of each phase space point moving along a classical orbit. Since phase space points within an invariant torus have same angular velocity, the shape of $F(\theta, n/2; t)$ within each individual torus does not change during time evolution. Thus the action-angle Wigner function in this time scale only changes the central point of $F(\theta, n/2; t)$ within each torus of the phase space, and nearby tori move against each other with a constant angular velocity ω_2. At time $t = \pi/(M\omega_2)$, the central points of nearby tori move against each other by π/M. Taking into account the symmetry of the action-angle Wigner function

$$F(\theta, n/2; t) = (-1)^n F(\theta + \pi, n/2; t) \tag{49}$$

one sees that an initial wavepacket now becomes M groups equally spaced in the angle direction. The shape of each group like a small wavepacket.

This process can also be described in terms of the Husimi distribution by decomposing the initial Husimi function into a superposition of small components [2]. Each component is the Husimi functions of the density operators

$$\hat{\rho}_{ij}^{(N)} = \sum_n |\phi_{n,i}\rangle\langle\phi_{n,i}|\hat{\rho}_{ij}(0)|\phi_{N-n,j}\rangle\langle\phi_{N-n,j}|. \tag{50}$$

From (4), one sees that the Husimi distribution of $\hat{\rho}_{ij}^{(N)}$ is similar to a small wavepacket with same symmetry as (49). Similar to (48), one can show that the evolution of a small component is like a phase space point: the central point of the component moves along a classical orbit and the shape does not change in the time scale $t \sim T_2$. Thus the partial revivals occur when nearby components move against each other by π/M.

The small components begin to spread when the third-order term of the spectrum's Taylor expansion takes effect on the phase shift $E(n, s)t/\hbar$. This makes the full and partial revivals disappear gradually at times $t = k_2\pi/\omega_2 \sim T_3$. But at times near $t = \frac{k_3}{3!}\frac{2\pi}{\omega_3}$ with k_3 being integer, all the components recover their initial forms. The phase shift $E(n, s)t/\hbar$ near these times can be equivalently treated as a quadratic function of the quantum number n. The overall effect of the evolution is equivalent to that of each component moving along a classical orbit. Thus the full and partial revivals near these times can be treated in the same way as in the time scale $t \sim T_2$. Figures 2(*a*) and (*b*) show the partial revivals in this time scale, which are one-to-one correspondent to figures 1(c) and (*d*). Figure 2(*c*) shows the full revival of the wavepacket in this time scale. The behaviour of the wavepacket in other time scales can be analysed in the same way.

Full and partial revivals can also be exhibited by expectation values of observables or by the autocorrelation function $P(t) = |\langle\Psi(0)|\Psi(t)\rangle|^2$, as shown in figure 3. In the short time scale $t \sim T_1 = 2\pi/\omega_1$, the classical motion of the wavepacket corresponds to the Rabi oscillations, which collapse when time reaches the time scale T_2. At times of partial revivals, the amplitude of oscillation increases to $1/M$ of the initial amplitude while the frequency of oscillation increases to M times of the initial value. This is a result of the fact that there are M small wavepackets passing through the initial point at times of partial revivals.

5. Conclusions

In summary, by means of the action-angle Wigner function, we have predicted a kind of equal partial revivals of wavepackets in various time scales. The basic idea is to decompose the

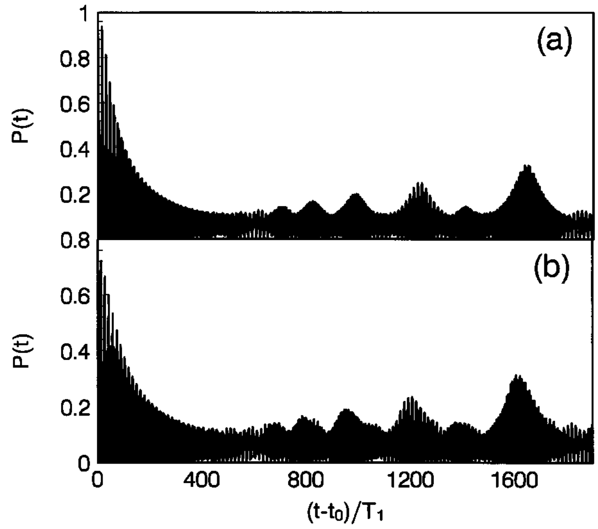

Figure 3. Autocorrelation function versus time. The initial time is set as (*a*) $t_0 = 0$, (*b*) $t_0 = 5.069\,79 \times 10^9$ which is the full revival time in the time scale $t \sim T_3$ as shown in figure 2(*c*).

density operator into a superposition of small components. Each small component behaves like a small wavepacket. In the action-angle phase space, a component is only distributed within one invariant torus, and never mixes with other components during evolution. The equal partial revivals occur when all small components recover their initial form, and their central points are equally spaced along the invariant tori of the phase space. We have obtained analytic expressions for one-dimensional cases. Our results show that, within a time scale T_2 near each full revival, there are such kind of equal partial revivals.

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Appendix

In this appendix, we give a simple proof of the theorem: If a real function $F(n)$ can be expressed as

$$F(n) - F(m) = f(n+m)(n-m) \pmod{2\pi} \tag{A.1}$$

for all possible integers n and m, then $f(n)$ must be a 'linear' function of integer n:

$$f(n) = \lambda n + \alpha \pmod{s\pi} \tag{A.2}$$

where α, λ are two constants, and $s = 1$ (for even number n) or 2 (for odd number n). Thus (A.1) becomes

$$F(n) - F(m) = [\lambda(n+m) + \alpha](n-m) \pmod{2\pi}. \tag{A.3}$$

Proof. Let a and $b(\neq a)$ be two fixed integers and x be an integer that keeps $(a+b+x)/2$ being an integer. From a, b, and x we construct three integers: $A = (a-b+x)/2$, $B = (a+b-x)/2$, $C = (-a+b+x)/2$, thus $a = A + B$, $b = B + C$, $x = C + A$. From (A.1), we have

$$F(B) - F(A) = f(a)(b-x) \pmod{2\pi} \tag{A.4}$$

$$F(C) - F(B) = f(b)(x-a) \pmod{2\pi} \tag{A.5}$$

$$F(C) - F(A) = f(x)(b-a) \pmod{2\pi}. \tag{A.6}$$

Using (A.4) and (A.5)−(A.6), we obtain

$$f(x)(b-a) = f(a)(b-x) + f(b)(x-a) \pmod{2\pi}. \tag{A.7}$$

For odd number x, putting $a = n_0$ (n_0 is an even number) and $b = n_0 + 1$ into (A.7), we have

$$f(x) = f(n_0)(n_0+1-x) + f(n_0+1)(x-n_0) \pmod{2\pi}. \tag{A.8}$$

For even number x, putting $a = n_0 + 1$ and $b = n_0 + 3$ into (A.7), we have

$$2f(x) = f(n_0+1)(n_0+3-x) + f(n_0+3)(x-n_0-1) \pmod{2\pi}. \tag{A.9}$$

From (A.8), $f(n_0+3)$ can be written as

$$f(n_0+3) = -2f(n_0) + 3f(n_0+1) \pmod{2\pi}. \tag{A.10}$$

Thus (A.9) becomes

$$2f(x) = 2f(n_0)(n_0+1-x) + 2f(n_0+1)(x-n_0) \pmod{2\pi} \tag{A.11}$$

or

$$f(x) = f(n_0)(n_0+1-x) + f(n_0+1)(x-n_0) \pmod{2\pi}. \tag{A.12}$$

From (A.8) and (A.12), we obtain (A.2), where $\lambda = [f(n_0+1) - f(n_0)]$ and $\alpha = (n_0+1)f(n_0) - n_0 f(n_0+1)$ are two constants.

Putting (A.2) into (A.1), we obtain (A.3) by noting that if $n+m$ is an even number, $n-m$ is also an even number, thus $(n-m)k\pi = 0 (\mathrm{mod}\, 2\pi)$ for any integer k. □

References

[1] Knospe O and Schmidt R 1996 *Phys. Rev.* A **54** 1154
[2] Kasperkovitz P and Peev M 1995 *Phys. Rev. Lett.* **75** 990
[3] Leichtle C, Averbuckh I S and Schleich W P 1996 *Phys. Rev. Lett.* **77** 3999
Leichtle C, Averbuckh I S and Schleich W P 1996 *Phys. Rev.* A **54** 5299
[4] Braunt P A and Savichev V I 1996 *J. Phys. B: At. Mol. Opt. Phys.* **29** L329
[5] Averbukh I Sh and Perelman N F 1989 *Phys. Lett.* A **139** 449
[6] Bluhm R and Kostelecky V A 1995 *Phys. Rev.* A **51** 4767
[7] Atakishiyev N M, Chumakov S M, Rivera A L and Wolf K B 1996 *Phys. Lett.* A **215** 128
[8] Alber G and Zoller P 1991 *Phys. Rep.* **199** 231
[9] Balazs N L and Jennings B K 1984 *Phys. Rep.* **104** 347
[10] Jensen R V 1992 *Nature* **355** 311
[11] Bialynicka-Birula Z 1968 *Phys. Rev.* **173** 1207
Dodonov V V, Malkin L A and Man'ko V I 1974 *Physica* **72** 597
Yurke B and Stoler D 1986 *Phys. Rev. Lett.* **57** 1055
Atakishiyev N M, Chumakov S M, Rivera A L and Wolf K B 1996 *Phys. Lett.* A **215** 128
[12] Boris S D, Brandt S, Dahmen H D, Stroh T and Larsen M L 1993 *Phys. Rev.* A **48** 2574
[13] Meekhof D M, Monroe C, King B E, Itano W M and Wineland D J 1996 *Phys. Rev. Lett.* **76** 1796
[14] Barddroff P J, Leichtle C, Schrade G and Schleich W P 1996 *Phys. Rev. Lett.* **77** 2198
[15] Eberly J H, Narozhny N B and Sanchez-Mondragon J 1980 *Phys. Rev. Lett.* **44** 1323
[16] Nauenberg M, Stroud C and Yeazell J 1994 *Sci. Am.* **270** 24
[17] Wigner E 1932 *Phys. Rev.* **40** 749
[18] Hillary M, O'Connell R F, Scully M O and Wigner E 1984 *Phys. Rep.* **106** 121
[19] Husimi K 1940 *Prog. Phys. Math. Soc. Japan* **22** 264
[20] Jie Quan-lin, Shun-Jin Wang and Lian-Fu Wei 1998 *Phys. Rev.* A **57** 3262
[21] Einstein A 1917 *Verh. Dtsch. Phys. Ges.* **19** 82
Brillouin L 1926 *J. Phys. Radium* **7** 353

Keller J B 1958 *Ann. Phys., NY* **4** 180
[22] Brif C and Ben-Aryeh Y 1994 *Phys. Rev.* A **50** 3505
[23] Vaccaro J A and Pegg D T 1990 *Phys. Rev.* A **41** 5156
[24] Vaccaro J A 1995 *Phys. Rev.* A **52** 3474
[25] Lynch R 1995 *Phys. Rep.* **256** 367
[26] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[27] Yoo H L and Eberly J H 1985 *Phys. Rep.* **118** 239
[28] Shore B W and Knight P L 1993 *J. Mod. Opt.* **40** 1195
[29] Guo G C and Zheng S B 1996 *Phys. Lett.* A **223** 332
[30] Chough Y T and Carmicheal H J 1996 *Phys. Rev.* A **54** 1709